Measure Theory with Ergodic Horizons
Lecture 16

Integration.

Given a measurable space $(X, \mathcal{B})$, we denote

$$L := L(X,\mathcal{B}) := \text{the set of all } \mathcal{B}\text{-measurable real-valued functions } X \to \overline{\mathbb{R}} := [-\infty, +\infty]$$

$$L^+ := L^+(X,\mathcal{B}) := \{ f \in L(X,\mathcal{B}) : f \geq 0 \}.$$

Note that $L(X,\mathcal{B})$ is a vector space closed under products and limits, while $L^+(X,\mathcal{B})$ is also closed under non-negative linear combinations, products and limits (in particular, under infinite sums).

Def. An integral on $L^+(X,\mathcal{B})$ is a ctbly additive non-negative linear functional $I : L^+(X,\mathcal{B}) \to \overline{\mathbb{R}}$, i.e.

(i) $I(\alpha f + \beta g) = \alpha \cdot I(f) + \beta I(g)$ for all $f, g \in L^+$ and all $\alpha, \beta \in \overline{\mathbb{R}}$ such that $\alpha f + \beta g \in L^+$.

(ii) $I(f) \geq 0$ for all $f \in L^+$, so $f \leq g \implies I(f) \leq I(g)$.

(iii) $I\left(\sum_{n \in \mathbb{N}} f_n\right) = \sum_{n \in \mathbb{N}} I(f_n)$ for all $f_n \in L^+$.

Observation. Every integral I on $L^+(X,\mathcal{B})$ defines a measure μ_I on $\mathcal{B}$ by

$$\mu_I(B) := I(\mathbb{1}_B).$$

Proof. $\mu_I(\emptyset) = I(0) = 0$ by linearity (i), and the ctbl additivity is by (iii). $\square$

We would like to show the converse of this:

Theorem. For every measure μ on $(X, \mathcal{B})$ there is a unique integral I_μ, called the *integral over μ*, such that $\mu_{I_\mu} = \mu$.

Remark. Every integral I on $L^+(X, \mathcal{B})$ uniquely determines a linear functional on a subspace of $L(X, \mathcal{B})$.

Proof. Indeed, every $f \in L(X, \mathcal{B})$ decomposes $f = f_+ - f_-$ by taking

$$f_-(x) := \begin{cases} -f(x) & \text{if } f(x) \leq 0 \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad f_+ := \begin{cases} f(x) & \text{if } f(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}.$$

So we are forced to define $I(f) := I(f_+) - I(f_-)$ and this is well-defined for those $f \in L(X, \mathcal{B})$ for which $I(f_+) < \infty$ and $I(f_-) < \infty$. $\square$

Okay now given a measure space $(X, \mathcal{B}, \mu)$, we define an integral I_μ on $L^+(X, \mathrm{Meas}_\mu)$. We will denote this integral by $\int f \, d\mu := \int f(x) \, d\mu(x) := I_\mu(f)$.

Def. A μ-measurable function $f: X \to \mathbb{R}$ is called *simple* if it is a (finite) linear combination of indicator functions of μ-measurable sets. For any such function f with a representation $f = \sum_{i=0}^{n} a_i \mathbb{1}_{B_i}$, we define

$$\int f \, d\mu := \sum_{i=0}^{n} a_i \mu(B_i).$$

HW: This is well-defined, i.e. does not depend on the choice of representation.

Prop. A μ-measurable function $f: X \to \mathbb{R}$ is simple if and only if $f(X) \subseteq \mathbb{R}$ is finite.

Proof. HW.

For a simple function $f: X \to \mathbb{R}$, let $f(X) = \{a_0, a_1, \dots, a_n\}$. Then

$$f = \sum_{i=0}^{n} a_i \cdot \mathbb{1}_{f^{-1}(a_i)}$$

is called the standard representation of f.

Prop. The integral on the vector space S of simple functions satisfies the following:

(i) Linearity: $\int (\alpha f + \beta g)\, d\mu = \alpha \int f\, d\mu + \beta \int g \, d\mu$ for all $f, g \in S$ and $\alpha, \beta \in \mathbb{R}$.

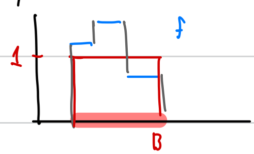
(ii) Non-negativity: $f \geq 0 \Rightarrow \int f \, d\mu \geq 0$, in particular, $f \leq g \Rightarrow \int f \, d\mu \leq \int g \, d\mu$.

(iii) $\int f \, d\mu = 0 \Rightarrow f = 0$ a.e.

(iv) If $f \geq 0$ then the function μ_f defined on all μ-measurable sets B by

$$\mu_f(B) := \int_B f \, d\mu := \int f \cdot \mathbb{1}_B \, d\mu$$

is a measure.



Proof. We only prove (iv). For this, we only have to show ctbl additivity. Let $B = \bigsqcup_{n \in \mathbb{N}} B_n$, and fix a representation $f = \sum_{i=0}^{m} a_i \mathbb{1}_{A_i}$. Then

$$\mu_f(B) = \int f \cdot \mathbb{1}_B \, d\mu = \int \sum_{i=0}^{m} a_i \mathbb{1}_{A_i \cap B} \, d\mu = \sum_{i=0}^{m} a_i \, \mu(A_i \cap B) \overset{\text{by ctbl additivity of } \mu}{=} \sum_{i=0}^{m} a_i \cdot \sum_{n \in \mathbb{N}} \mu(A_i \cap B_n)$$

$$= \sum_{n \in \mathbb{N}} \sum_{i=0}^{m} a_i \, \mu(A_i \cap B_n) = \sum_{n \in \mathbb{N}} \int \sum_{i=0}^{m} a_i \mathbb{1}_{A_i} \cdot \mathbb{1}_{B_n} \, d\mu = \sum_{n \in \mathbb{N}} \int_{B_n} f \, d\mu = \sum_{n \in \mathbb{N}} \mu_f(B_n). \quad \square$$

Having the integral defined for simple functions, we may define it for all of $L^+(X, \mathrm{Meas}_\mu)$ as follows: For $f \in L^+(X, \mathrm{Meas}_\mu)$:

$$\int f\,d\mu := \sup\left\{ \int g\,d\mu : g \in L^+(X, \mathrm{Meas}_\mu) \text{ is a simple function} \right\}.$$

This definition is only motivated if we prove that we can approximate every $f \in L^+$ with simple functions from below.

Prop. (a) For each $f \in L^+(X, \mu)$ there is an increasing sequence (f_n) of non-negative simple functions $\le f$ such that $f_n \nearrow f$, i.e. $\lim_{n\to\infty} f_n = f$ and $f_0 \le f_1 \le \dots$. Moreover, the convergence is uniform on every set $X' \subseteq X$ on which f is bounded, i.e.

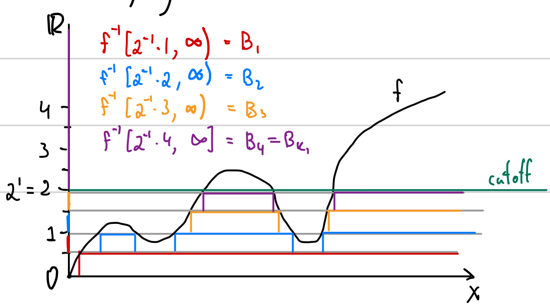
$$\| f_n|_{X'} - f|_{X'} \|_u \to 0 \quad \text{as } n \to \infty,$$

where $\|f - g\|_u := \sup_{x \in X} |f(x) - g(x)|$.

(b) For each $f \in L(X, \mu)$ there is a sequence (f_n) of simple functions such that $|f_0| \le |f_1| \le \dots \le |f|$ and $f_n \to f$. Moreover, the convergence uniform on every set $X' \subseteq X$ on which f is bounded.

Proof. (b) follows from (a) by getting increasing sequences (f_n^-) and (f_n^+) of non-negative simple functions approximating f_- and f_+ as in (a), then verifying that $f_n := f_n^+ - f_n^-$ is as desired.

(a) To define f_n, we "cutoff" the range of f at 2^n and split the rest $[0, 2^n]$ into



equal pieces of size 2^{-n}, i.e. into 2^{2n} pieces. Let $A_k := f^{-1}((2^{-n}\cdot k, 2^{-n}\cdot(k+1)])$ and set $f_n := \sum_{k=0}^{2^{2n}-1} 2^{-n}\cdot k \cdot \mathbb{1}_{A_k} = \sum_{k=1}^{2^{2n}} 2^{-n}\cdot \mathbb{1}_{B_k}$, where $B_k := f^{-1}([2^{-n}\cdot k, \infty))$, so $B_1 \supseteq B_2 \supseteq B_3 \supseteq \dots$.
Note that $A_k = B_{k+1} \setminus B_k$ for all $k < 2^{2n}$. Note that on $X_n := f^{-1}([0, 2^n])$, we have $\| f_n|_{X_n} - f|_{X_n} \|_u \leq 2^{-n}$. Since $X_n \nearrow X$, i.e. $X_0 \subseteq X_1 \subseteq X_2 \subseteq \dots$ and $\bigcup_{n\in\mathbb{N}} X_n = X$, we get that for each $x \in X$, $\lim_{n\to\infty} f_n(x) = f(x)$. Moreover, if f is bounded on a set $X' \subseteq X$, then $X' \subseteq X_m$ for some $m \in \mathbb{N}$, and for all $n \geq m$, we have $\| f_n|_{X_m} - f|_{X_m} \| \leq 2^{-n}$, so the convergence is uniform on X_m.
Lastly, note that if $f(x) = \infty$ then by construction, $f_n(x) = 2^n \to \infty = f(x)$. $\square$